Discontinuous Galerkin Methods

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A short (and biased) historical overview of the DG methods

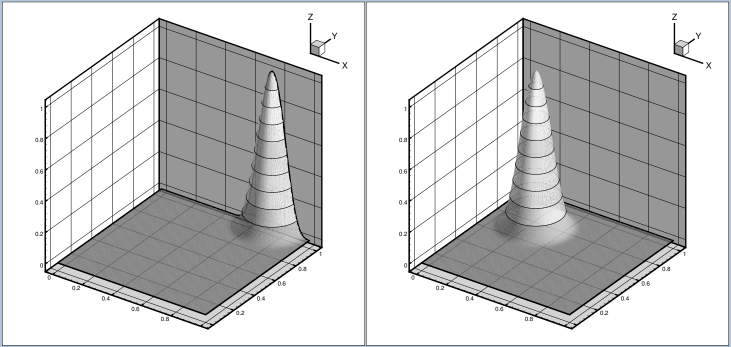
- First DG method introduced in 1973 by Reed and Hill for linear transport. First studied in 1974 by Lesaint and Raviart.
- Extended to nonlinear hyperbolic conservation laws in the 90's by B.C. and C.-W. Shu.
- Extended to compressible flow in 1997 first by F. Bassi and S. Rebay.
- New DG methods for diffusion appear and some old ones (the IP methods of the late 70's) are resuscitated. A unified analysis is proposed in 2002 by D. Arnold, F. Brezzi, B.C. and D. Marini.
- Explosive extension to a wide variety of equations.
- They *clash* with the well-established mixed and continuous Galerkin methods. In response, the HDG methods are introduced in 2009 by B.C., J. Gopalakrishnan and R. Lazarov. The HDG methods are strongly related to the hybrid methods and to the hybridization techniques of the mid 60's introduced as implementation techniques for mixed methods.

A short (and biased) historical overview of the DG methods

- B.C., G. Karniadakis, C.-W. Shu, *The development of Discontinuous Galerkin methods*, in Discontinuous Galerkin methods. Theory, computation and applications, Lecture Notes in Computational Sicence and Engineering, Volume 11, Springer, 2000.
- B.C. and C.-W. Shu, *Runge-Kutta Discontinuous Galerkin methods for convection-dominated problems*, J. Sci. Comput. 16 (2001), pp. 173–261.
- D. Arnold, F. Brezzi, B.C. and D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SINUM 39 (2002), pp. 1749–1779.
- B.C., *Discontinuous Galerkin methods*, ZAMM Z. Angew. Math. Mech. 83 (2003), pp. 731–754.
- B.C., *Discontinuous Galerkin methods for Computational Fluid Dynamics*, Encyclopedia of Computational Mechanics, Volume 3: Fluids, E. Stein, R. de Borst and T.J.R. Hughes, Eds., Wiley, 2004, pp. 91—123.

Motivation

Why use DG methods? Good approximation of smooth solutions.

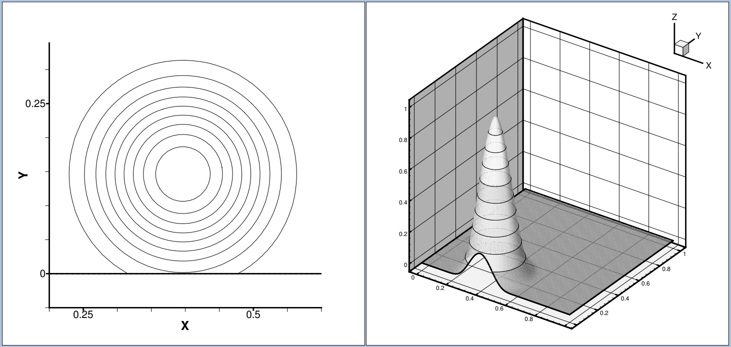


Approximate solution at $T = 0.0$ (left), $T = \frac{3}{8}\pi$ (right) with quadratic polynomials.

(B.C. and C.-W. Shu, 1990.)

Motivation

Why use DG methods? Good approximation of smooth solutions.

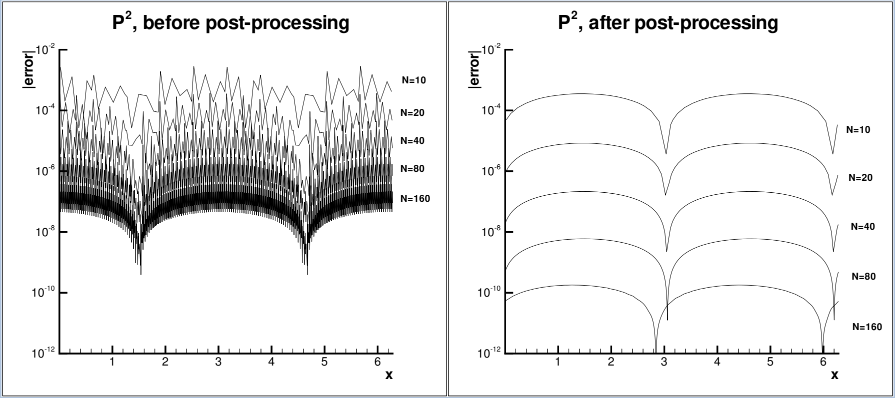


Approximate solution at $T = \frac{3}{4}\pi$ with quadratic polynomials.

(B.C. and C.-W. Shu, 1990.)

Motivation

Why use DG methods? Local postprocessing enhances the accuracy.

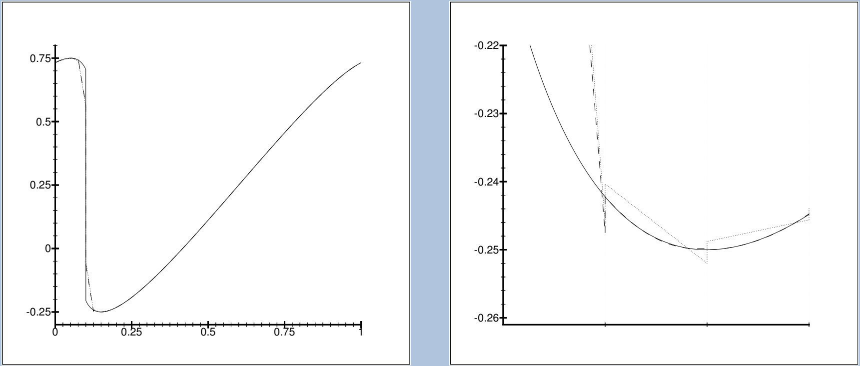


The absolute value of the errors for P^2 with N=10, 20, 40, 40, 80 and 160 elements. Before post-processing (left) and after post-processing (right).

(B.C., M. Luskin, C.-W. Shu and E. Suli, 2003).

Motivation

Why use DG methods? Good approximation of discontinuities.

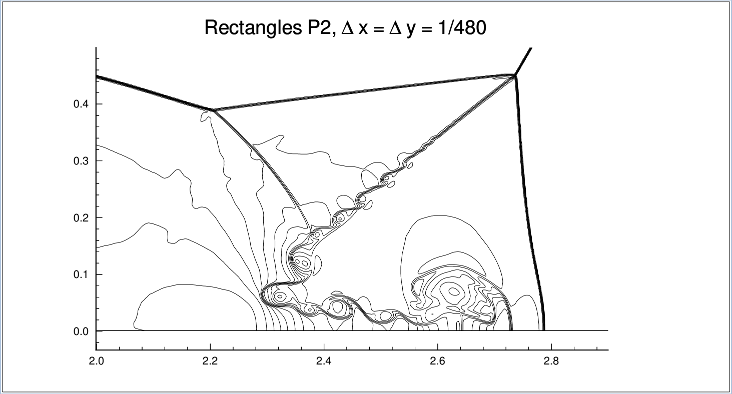


Inviscid Burgers' equation.

(B.C. and C.-W. Shu, 1990).

Motivation

Why use DG methods? Good approximation of contacts and shocks.

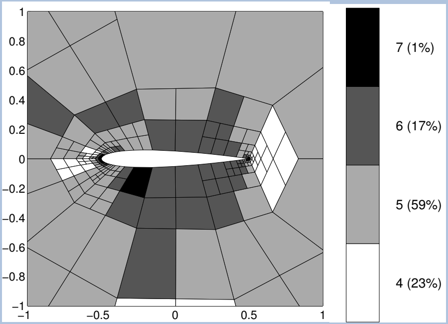


Isolines of the density for the Double Mach reflection problem.

(B.C. and C.-W. Shu, 1998).

Motivation

Why use DG methods? Ideally suited for adaptivity.

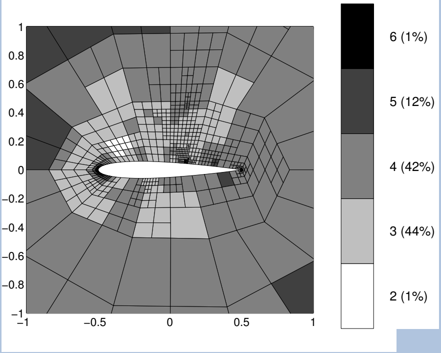


Subsonic flow around a NACA0012 airfoil. The *hp*–mesh has 325 elements, 45008 degrees of freedom, and produces an error $|J(\mathbf{u}) - J(\mathbf{u}_h)| = 3.756 \times 10^{-7}$.

(Houston and Suli, 2002).

Motivation

Why use DG methods? Ideally suited for adaptivity.



Supersonic flow around a NACA0012 airfoil: The *hp*–mesh has 783 elements, 69956 degrees of freedom, and produces an error of $|J(\mathbf{u}) - J(\mathbf{u}_{\mathrm{DG}})| = 1.311 \times 10^{-4}$.

(Houston and Suli, 2002).

The original DG method.

Transport of neutrons.

The original DG method was devised for numerically solving equations modeling the *transport of neutrons*. A simplified version of that model is the following:

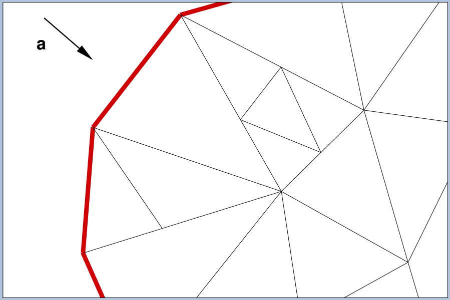
$$\begin{aligned} \sigma\, u + \nabla \cdot (\boldsymbol{a}\, u) &= f && \text{in } \Omega, \\ u &= u_D && \text{on } \partial\Omega_-, \end{aligned}$$

where $\sigma > 0$, $\boldsymbol{a}$ is a constant vector and $\partial\Omega_-$ the *inflow* boundary of $\Omega \subset \mathbb{R}^d$, that is, $\partial\Omega_- = \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{a} \cdot \boldsymbol{n}(\boldsymbol{x}) < 0\}$.

The original DG method

Transport of neutrons.

Triangulation $\Omega_h = \{K\}$ of Ω and boundary data u_D on $\partial\Omega_-$.



The original DG method.

Rewriting the equations.

- Set $\widehat{u} := u_D$ on $\partial\Omega_-$.
- Given $\widehat{u}$ on ∂K_-, compute u by solving

$$\begin{aligned} \sigma\, u + \nabla\cdot(\boldsymbol{a}\, u) &= f && \text{in } K,\\ u &= \widehat{u} && \text{on } \partial K_-. \end{aligned}$$

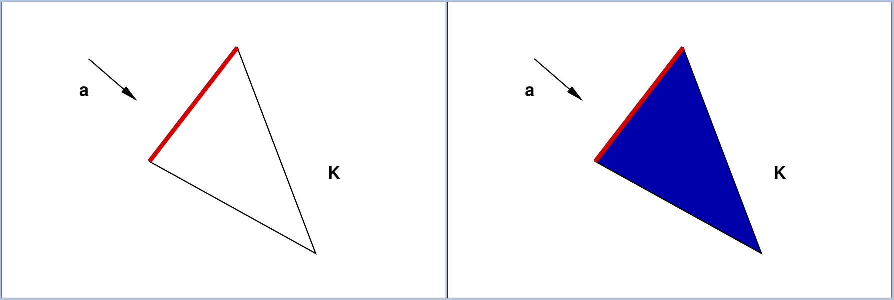
- Given u in K, set $\widehat{u} := u$ on $\partial K \setminus \partial K_-$.

Here $\partial K_- := \{\boldsymbol{x} \in \partial K : \boldsymbol{a}\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$.

The original DG method

Solving the equations.

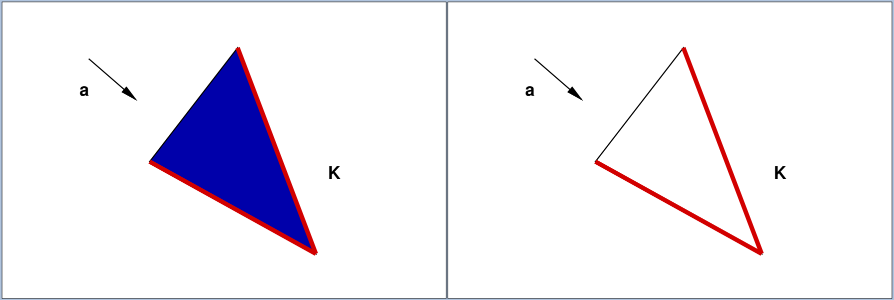
Given $\widehat{u}$ on ∂K_- (left), compute u on K (right).



The original DG method

Solving the equations.

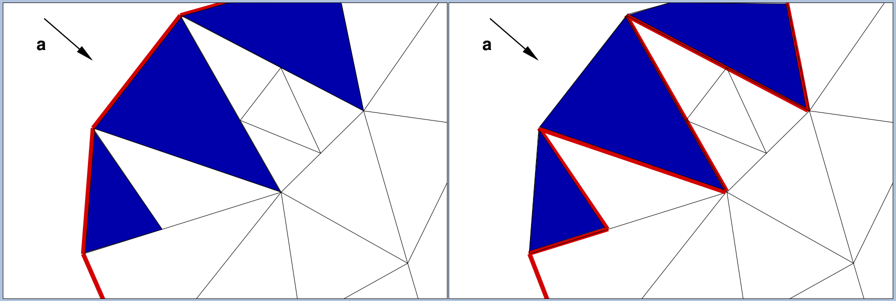
Set $\widehat{u} := u$ on $\partial K \setminus \partial K_-$ (left). The computation on other elements can now proceed (right).



The original DG method

Solving the equations.

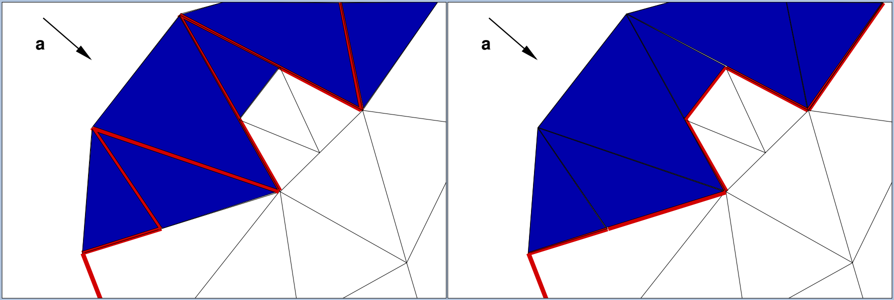
Given $\color{red}{\widehat{u}} := u_D$ on $\partial\Omega_-$, compute $\color{blue}{u}$ (left) and then obtain $\color{red}{\widehat{u}}$ (right).



The original DG method

Solving the equations.

Given $\color{red}{\widehat{u}}$, compute $\color{blue}{u}$ (left) and then obtain $\color{red}{\widehat{u}}$ (right).



The original DG method

The weak formulation on each element.

Given $\widehat{u}$ on ∂K_-, we have that u satisfies the weak formulation

$$\begin{aligned}\sigma\,(u, w)_K - (u, \boldsymbol{a}\cdot\nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n}u, w\rangle_{\partial K\setminus\partial K_-} \\ = (f, w)_K - \langle \boldsymbol{a}\cdot\boldsymbol{n}\widehat{u}, w\rangle_{\partial K_-}\,,\end{aligned}$$

for all $w \in W(K)$.

The original DG method

The Galerkin method on each element.

The Galerkin method on the element $K \in \Omega_h$ is defined as follows. We take u_h in the space $W(K)$ and determine it by requiring that

$$\begin{aligned}\sigma\,(u_h, w)_K - (u_h, \boldsymbol{a}\cdot\nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n}\,u_h, w\rangle_{\partial K\setminus\partial K_-} \\ = (f, w)_K - \langle \boldsymbol{a}\cdot\boldsymbol{n}\,\widehat{u}_h, w\rangle_{\partial K_-}\,,\end{aligned}$$

for all $w \in W(K)$.

The original DG method

Implementation.

- Set $\widehat{u}_h := u_D$ on $\partial\Omega_-$.
- Given $\widehat{u}_h$ on ∂K_-, compute u_h in K as the element of $W(K)$ such that

$$\begin{aligned}\sigma\,(u_h, w)_K - (\boldsymbol{a} u_h, \nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n} u_h, w\rangle_{\partial K\setminus\partial K_-} \\ = (f, w)_K - \langle \boldsymbol{a}\cdot\boldsymbol{n}\widehat{u}_h, w\rangle_{\partial K_-}\,,\end{aligned}$$

  for all $w \in W(K)$.
- Given u_h in K, set $\widehat{u}_h := u_h$ on $\partial K \setminus \partial K_-$.

The original DG method.

The stabilization mechanism. The jumps $u_h - \widehat{u}_h$ stabilize the method.

The energy identity for the exact solution is

$$\sigma \| u - f/2\sigma \|^2_{L^2(\Omega)} + \frac{1}{2} \langle |\boldsymbol{a} \cdot \boldsymbol{n}| u, u \rangle_{\partial\Omega_+} = \Psi(f, u_D),$$

and for the approximate solution,

$$\sigma \| u_h - f/2\sigma \|^2_{L^2(\Omega)} + \frac{1}{2} \langle |\boldsymbol{a} \cdot \boldsymbol{n}| u_h, u_h \rangle_{\partial\Omega_+} + \Theta_h(u_h - \widehat{u}_h) = \Psi(f, u_D),$$

where $\Theta_h(u_h - \widehat{u}_h) := \frac{1}{2} \sum_{K \in \Omega_h} \langle |\boldsymbol{a} \cdot \boldsymbol{n}| (u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial K_-}$.

The method is *stabilized* by the term $\Theta_h(u_h - \widehat{u}_h)$.

The original DG method.

The stabilization mechanism. The jumps $u_h - \widehat{u}_h$ control the residuals.

The Galerkin formulation on the element K reads

$$\sigma\,(u_h, w)_K - (u_h, \boldsymbol{a}\cdot\nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n}\,\widehat{u}_h, w\rangle_{\partial K} = (f, w)_K \quad \forall\, w \in W(K),$$

or, equivalently,

$$(R_K, w)_K = \langle R_{\partial K}, w\rangle_{\partial K} \quad \forall\, w \in W(K),$$

where $R_K := \sigma\, u_h + \nabla\cdot(\boldsymbol{a}\, u_h) - f$ and $R_{\partial K} := \boldsymbol{a}\cdot\boldsymbol{n}\,(u_h - \widehat{u}_h)$.

Thus, the L^2-projection of R_K into $W(K)$ is controlled by the jumps $R_{\partial K} = \boldsymbol{a}\cdot\boldsymbol{n}\,(u_h - \widehat{u}_h)$.

The original DG method.

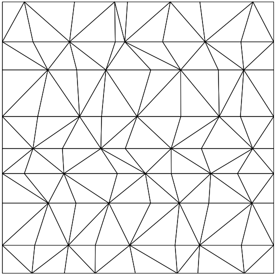
The stabilization mechanism. The case of non-smooth solutions.

- The exact solution u in the element K is not smooth.
- The residual R_K is big.
- The jump $R_{\partial K} = |\boldsymbol{a} \cdot \boldsymbol{n}|(u_h - \widehat{u}_h)$ is big.
- The dissipation produced by $\Theta_h(u_h - \widehat{u}_h)$ damps the spurious oscillations.

The original DG method.

Convergence properties. The spaces and the triangulations.

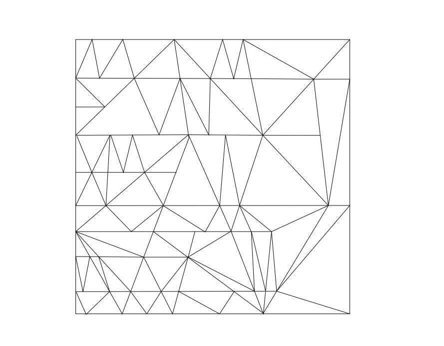
- special triangulations Ω_h made of shape-regular simplexes K,
- $W(K) := \mathcal{P}_k(K)$,



A special triangulation for $\boldsymbol{a} = (1, 0)$.

The original DG method.

Convergence properties. Another triangulation.



Another special triangulation for $\boldsymbol{a} = (1,0)$.

The original DG method.

Convergence properties. The auxiliary projection.

We can find projections such that the projection of the errors

- $\Pi : H^1(K) \to W(K)$, $\qquad \varepsilon_u := \Pi(u - u_h)$,
- $\mathsf{P}_\partial : L^2(F) \to M(F)$, $\qquad \varepsilon_{\widehat{u}} := \mathsf{P}_\partial(u - \widehat{u}_h)$,

satisfy

- $\boldsymbol{a} \cdot \boldsymbol{n}\, \varepsilon_{\widehat{u}} = \boldsymbol{a} \cdot \boldsymbol{n}\, \widehat{\varepsilon_u}$,
- for all $w \in W(K)$,

$$\sigma\,(\varepsilon_u, w)_K - (\varepsilon_u, \boldsymbol{a} \cdot \nabla w)_K + \langle \boldsymbol{a} \cdot \boldsymbol{n}\, \widehat{\varepsilon_u}, w \rangle_{\partial K} = \sigma\,(\Pi u - u, w)_K,$$

- $\widehat{\varepsilon_u} = 0$ on $\partial\Omega_-$.

From the energy identity

$$\sigma \|\varepsilon_u - \frac{1}{2}(\Pi u - u)\|^2_{L^2(\Omega)} + \Theta_h(\varepsilon_u - \widehat{\varepsilon_u}) = \frac{\sigma}{4}\|\Pi u - u\|^2_{L^2(\Omega)},$$

we deduce that

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)} + \sigma^{-1/2}\Theta_h^{1/2}(\varepsilon_u - \widehat{\varepsilon_u}) &\le C\,\|\Pi u - u\|_{L^2(\Omega)} \\ &\le C\,|\,u\,|_{H^{k+1}(\Omega_h)}\,h^{k+1}. \end{aligned}$$

Thus, optimal convergence orders are obtained for smooth solutions.

(B.C., B. Dong and J. Guzmán, 2008.)

The original DG method.

Conclusion.

We have seen that the original DG method:

- Uses discontinuous approximations for both the solution inside each element and its trace on the element boundary.
- Uses a Galerkin method to weakly enforce the equations on each element.
- Is devised so that they can be efficiently implemented.
- Has a stabilization mechanism that allows it to damp away spurious oscillations and reach optimal orders of convergence at the same time.

DG methods for linear symmetric hyperbolic systems

$$
\begin{aligned}
\boldsymbol{u}_t + \nabla\cdot \mathrm{F}(\boldsymbol{u}) + B u &= \boldsymbol{f} && \text{in } \Omega\times(0,T),\\
\boldsymbol{u} &= \boldsymbol{g} && \text{at } t=0,\\
\mathrm{F}(\boldsymbol{u})\,\boldsymbol{n} - \mathcal{N}\boldsymbol{u} &= 0 && \text{on } \partial\Omega\times[0,T].
\end{aligned}
$$

Here $(\mathrm{F}(\boldsymbol{u}))_{ij} := \sum_{\ell=1}^{m}(A_j)_{i\ell}u_\ell$, and A_j, $j=1,\ldots,N$, are constant, symmetric matrices.

In 1958, Friedrichs showed that the above problem has a unique solution if

$$\begin{aligned} &\mathcal{N} + \mathcal{N}^\star \geq 0, \\ &B + B^\star \geq \sigma\, Id, \qquad \sigma \geq 0, \\ &ker(A_n - \mathcal{N}) + ker(A_n + \mathcal{N}) = \mathbb{R}^m. \end{aligned}$$

Here $A_n := \sum_{i=1}^{N} A_j\, n_j$. Note that $\mathrm{F}(\boldsymbol{u})\, \boldsymbol{n} = A_n \boldsymbol{u}$.

$$\rho\frac{\partial^2 u}{\partial t^2} - \nabla\cdot(\mathrm{A}\nabla u) = f \quad \text{in } \Omega\times(0,T).$$

$$\begin{aligned}
\mathrm{c}\frac{\partial \boldsymbol{q}}{\partial t} - \nabla v \quad &= \quad 0 \qquad \text{in } \Omega\times(0,T),\\
\rho\frac{\partial v}{\partial t} - \nabla\cdot\boldsymbol{q} \quad &= \quad f \qquad \text{in } \Omega\times(0,T).
\end{aligned}$$

DG methods for linear symmetric hyperbolic systems

Elastodynamics: The first-order system

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \nabla \cdot [\mu \nabla \boldsymbol{u} + (\mu + \lambda)(\nabla \cdot \boldsymbol{u})\mathbf{I}] \quad = \quad \boldsymbol{b} \qquad \text{in } \Omega \times (0, T).$$

$$\begin{aligned}
\frac{\partial \mathbf{H}}{\partial t} - \nabla \boldsymbol{v} &= 0 && \text{in } \Omega \times (0, T), \\
\rho \frac{\partial \boldsymbol{v}}{\partial t} - \nabla \cdot (\mu \mathbf{H} + p\mathbf{I}) &= \boldsymbol{b} && \text{in } \Omega \times (0, T), \\
\epsilon \frac{\partial p}{\partial t} - \nabla \cdot \boldsymbol{v} &= 0 && \text{in } \Omega \times (0, T).
\end{aligned}$$

DG methods for linear symmetric hyperbolic systems

Maxwell's equations

$$\mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0,$$
$$\epsilon \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = 0,$$

We take $\boldsymbol{u}_h(t)$ in the space $\boldsymbol{W}(K)$ and determine it by requiring that

$$((\boldsymbol{u}_h)_t, \boldsymbol{w})_K - (\mathrm{F}(\boldsymbol{u}_h), \nabla \boldsymbol{w})_K + \left\langle \widehat{\mathrm{F}_h \boldsymbol{n}}, \boldsymbol{w} \right\rangle_{\partial K} = (\boldsymbol{f}, \boldsymbol{w})_K,$$

for all $\boldsymbol{w} \in \boldsymbol{W}(K)$, where

$$\widehat{\mathrm{F}_h \boldsymbol{n}} := A_{n^+}(\frac{1}{2}(\boldsymbol{u}_h^+ + \boldsymbol{u}_h^-)) + \frac{1}{2}\mathcal{N}_{n^\pm}(\boldsymbol{u}_h^+ - \boldsymbol{u}_h^-)$$

The matrix $\mathcal{N}_{n^\pm}$ is called the dissipation matrix.

• In the scalar case, $\mathrm{F}(\boldsymbol{u}) = \boldsymbol{a}\, u$. For the original DG method: $\widehat{\mathrm{F}_h \boldsymbol{n}}$ is the upwinding numerical flux $A_{n^+} := \boldsymbol{a} \cdot \boldsymbol{n}^+$ and $\mathcal{N}_{n^\pm} = |\boldsymbol{a} \cdot \boldsymbol{n}|$.

• The upwinding numerical flux $\widehat{\mathrm{F}_h \boldsymbol{n}}$ in the general case is obtained as follows:

- Diagonalize $A_n = P^{-1}\,\Lambda\, P$,
- Set $\mathcal{N}_{n^\pm} := P^{-1}\,|\,\Lambda\,|\,P$.

• The Lax-Friedrichs numerical flux $\widehat{\mathrm{F}_h \boldsymbol{n}}$ is obtained as follows:

- Diagonalize $A_n = P^{-1}\,\Lambda\, P$,
- Set $\mathcal{N}_{n^\pm} := \lambda_{\max}\mathrm{Id}$.

- The jumps $\boldsymbol{u}_h{}^+ - \boldsymbol{u}_h{}^-$ stabilize the method when $\mathcal{N}_{\boldsymbol{n}^\pm}$ is positive definite.
- The jumps control the residuals.
- Spurious oscillations are damped in the presence of discontinuities.
- The method converges with order $\mathrm{k}+1/2$.
- After a local postprocessing, with order $2k+1$ for locally uniform grids.

Time discretization of linear symmetric hyperbolic systems

Main strategies.

- Explicit Runge-Kutta methods: SSP methods. (C.-W. Shu 88, S. Gottlieb, C.-W. Shu and E. Tadmor, 2001.)
- Space-time methods: Locally implicit. (R. Haber; J. van der Vegt; R. Falk and G. Richter, 1999; see also, Gopalakrishnan, Schöberl and Winterstiger, 1917)
- Globally implicit methods: Efficient multigrid techniques. (J. van der Vegt; P. Persson and J. Peraire.)

The original DG method.

Dispersion and dissipation properties.

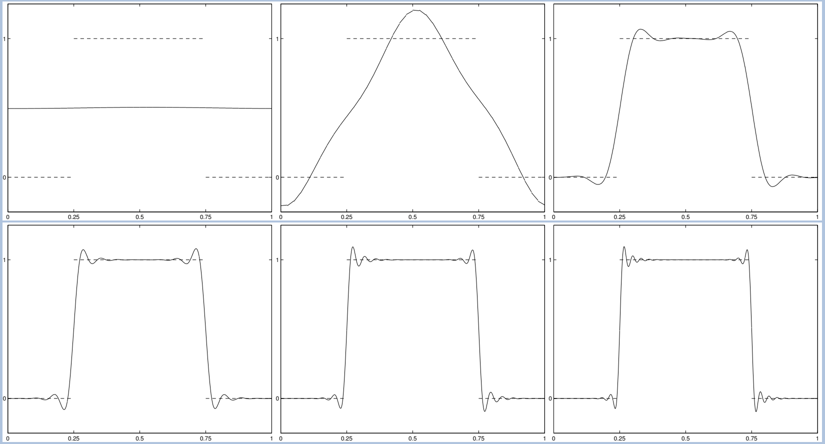
For the semidiscrete transport equation:

- Order of dispersion: $2\,k + 3$.
- Order of dissipation: $2\,k + 2$.

(M. Ainsworth, 2004).

The original DG method.

Dispersion and dissipation properties.



Effect of the polynomial degree on the approximation of discontinuities.

(B.C. and C.-W. Shu, 2001.)

We have devised DG methods that:

- Use discontinuous approximations for both the solution inside each element and its trace on the element boundary.
- Use a Galerkin method to weakly enforce the equations on each element.
- Have a stabilization mechanism that allows it to damp away spurious oscillations and reach almost optimal orders of convergence at the same time.

$$\mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0.$$

Hyperbolic: $\sum_{i=1}^{d} \frac{\partial \mathbf{f}_i}{\partial \mathbf{u}}(\mathbf{u})\, n_i$ is diagonalizable and has real eigenvalues.

Example 1: The Euler equations of gas dynamics:

$$\begin{aligned}
&\rho_t + (\rho\, v_j)_{,j} = 0,\\
&(\rho\, v_i)_t + (\rho\, v_i\, v_j - \sigma_{ij})_{,j} = f_i,\\
&(\rho\, e)_t + (\rho\, e\, v_j - \sigma_{ij}\, v_i)_{,j} = f_i\, v_i,
\end{aligned}$$

where $\sigma_{ij} = -p\,\delta_{ij}$ and $e = \frac{p}{(\gamma-1)\,\rho} + \frac{1}{2}\,|\,\mathbf{v}\,|^2$.

Example 2: Isentropic gas dynamics in Lagrangian coordinates,

$$\tau_t - u_x = 0,$$
$$u_t + (p(\tau))_x = 0,$$

and in Eulerian coordinates

$$\rho_t + (v\,\rho)_x = 0,$$
$$(\rho\,v)_t + (\rho\,v^2 + p(\rho^{-1}))_x = 0,$$

where $p(\tau) = A\tau^{-\gamma}$ for a polytropic ideal gas.

Example 3: Scalar hyperbolic conservation law:

$$\mathrm{u}_t + \nabla \cdot \mathbf{f}(\mathrm{u}) = 0.$$

Inviscid Burgers equation: 1D and $\mathbf{f}(\mathrm{u}) = \mathrm{u}^2/2$.

The RKDG methods.

Non-linear hyperbolic problems.

Main difficulties:

- Convergence to the physically relevant solution must be ensured.
- An additional mechanism to properly capture discontinuities is needed.
- Implicit methods are very inefficient in the presence of discontinuities.

Solution:

- **DG-space discretization** with suitable **numerical traces** (approximate Riemann solvers).
- **SSP, explicit** time-marching algorithms.
- **Slope limiters** (part of an artificial viscosity hidden term!).

Examples of numerical fluxes:

- The Godunov flux:

$$\begin{aligned} \widehat{f}(a,b) &= \min_{a\le u\le b} f(u), && \text{if } a\le b,\\ \widehat{f}(a,b) &= \max_{b\le u\le a} f(u), && \text{otherwise.} \end{aligned}$$

- The Engquist-Osher flux:

$$\begin{aligned} \widehat{f}(a,b) &= \int_0^b \min(f'(s),0)\, ds \\ &\quad + \int_0^a \max(f'(s),0)\, ds + f(0). \end{aligned}$$

- The Lax-Friedrichs flux:

$$\widehat{f}(a,b) = \frac{1}{2}\,[f(a)+f(b)-C\,(b-a)],$$

$$C = \max_{\inf u^0(x)\le s\le \sup u^0(x)} |f'(s)|.$$

Development of the RKDG method:

- 1982: G.Chavent and G.Salzano: Use the **DG-space discretization** with Godunov flux.
- 1989: G.Chavent and B.C.: Incorporate the **slope limiter**.
- 1991: B.C. and C.-W.Shu: Incorporate an **SSP time-marching method**: First RKDG method.
- 89-98: B.C. and C.-W.Shu (+S.Hou+S.Lin) : RKDG methods.

A parallel development:

- 1987: Allmaras and Giles: Euler equations.
- 1989: Allmaras: P^1 and 3-stage second-order RK.
- 1991: Halt and Agarwall
- 1992: Halt: high polynomial degree.

The RKDG method

We construct the RKDG methods for the non-linear hyperbolic model problem

$$\begin{aligned} u_t + f(u)_x &= 0, && \text{in } (0,1) \times (0,T), \\ u(\cdot,0) &= u_0(\cdot) && \text{on } (0,1), \\ u(0+,\cdot) &= u(1-,\cdot) && \text{on } (0,T), \end{aligned}$$

The main components of the RKDG methods are:

- A DG space discretization,
- A strongly-stable RK time-marching discretization,
- A generalized slope limiter,

- <u>Discontinuous Galerkin discretization in space</u>

The approximate solution u_h restricted to the interval I_j belongs to the space $P(I_j)$.

The non-linear conservation law element-by-element by requiring that for every function v_h in the space $P(I_j)$

$$((u_h)_t, v)_{I_j} - (f(u_h), (v)_x)_{I_j} + \widehat{f}(u_h)\, v\Big|_{x_{j-1/2}}^{x_{j+1/2}} = 0,$$

where $\widehat{f}(u_h)$ is the so-called **numerical flux** has the following general form:

$$\widehat{f}(u_h)(x_{j+1/2}) = \widehat{f}(u_h(x_{j+1/2}^-), u_h(x_{j+1/2}^+)).$$

Monotone schemes are obtained with $k = 0$.

• Strong-Stability-Preserving RK methods

Each time step for $\frac{d}{dt} u_h = L(u_h)$ is of the form

1. set $u_h^{(0)} = u_h^n$;
2. for $i = 1, ..., K$ compute the intermediate functions:
$$u_h^{(i)} = \sum_{l=0}^{i-1} \alpha_{il}\, w_h^l,$$
$$w_h^l = u_h^{(l)} + \frac{\beta_{il}}{\alpha_{il}} \Delta t^n L_h(u_h^{(l)});$$
3. set $u_h^{n+1} = u_h^K$.

Note that $\alpha_{il} \in [0,1]$, and that, if $\alpha_{il} = 0$, then $\beta_{il} = 0$.

Set

$$w_h = u_h + \delta L_h(u_h) \equiv \mathrm{EULER}(u_h;\delta),$$

and assume that

$$|\, w_h \,| \le |\, u_h \,| \qquad \forall |\, \delta \,| \le \delta_0.$$

Then

$$|\, u_h^{(i)} \,| \le \sum_{l=0}^{i-1} \alpha_{il} \,|\, w_h^l \,| \le \sum_{l=0}^{i-1} \alpha_{il} \,|\, u_h^{(l)} \,|,$$

provided that

$$\frac{\beta_{il}}{\alpha_{il}} \Delta t^n \le \delta_0.$$

This implies that

$$|\, u_h^n \,| \le |\, u_h^0 \,| \qquad \forall n = 0, \ldots, N.$$

The Euler step is non-increasing in $|\cdot|$ if:

- We take the semi-norm $|\cdot|$ to be

$$|\,u_h\,| \equiv \sum_j |\,\overline{u}_{j+1} - \overline{u}_j\,|,$$

where $\overline{u}_j = \frac{1}{\Delta_j}\int_{I_j} u_h(x)\,dx$.

- We take

$$\delta_0^{-1} = 2\,\left(\frac{|\,\widehat{f}(a,\cdot)\,|_{Lip}}{\Delta_{j+1}} + \frac{|\,\widehat{f}(\cdot,b)\,|_{Lip}}{\Delta_j}\right).$$

- We assume that the following **sign** conditions are satisfied:

$$\begin{aligned}\operatorname{sign}(u^+_{j+1/2} - u^+_{j-1/2}) &= \operatorname{sign}(\overline{u}_{j+1} - \overline{u}_j),\\ \operatorname{sign}(u^-_{j+1/2} - u^-_{j-1/2}) &= \operatorname{sign}(\overline{u}_j - \overline{u}_{j-1}).\end{aligned}$$

•The generalized slope limiter

Since the **sign** conditions are not automatically satisfied, we **enforce** them by means of a simple projection called the generalized slope limiter, $\Lambda\Pi_h$.

It is indeed possible to construct generalized slope limiters that enforce the sign conditions which, moreover, have the following properties:

- Is a projection into the finite element space.
- Leaves the averages unchanged.
- Leaves a linear function unchanged.
- Can be efficiently parallelized.

The slope limiter of the MUSCL scheme is the prototypical example.

•The RKDG method

- Set $u_h^0 = \Lambda\Pi_h P_h u_0$.
- For $n = 0$ until $N-1$ do:

1. set $u_h^{(0)} = u_h^n$;
2. for $i = 1, \ldots, K$ compute:

$$u_h^{(i)} = \Lambda\Pi_h \left(\sum_{l=0}^{i-1} \alpha_{il}\, w_h^l \right),$$

$$w_h^l = \mathrm{EULER}(u_h^{(l)}; \frac{\beta_{il}}{\alpha_{ij}} \Delta t^n);$$

3. set $u_h^{n+1} = u_h^K$.

We have the following boundedness result.

Theorem

Assume that

$$\frac{\beta_{il}}{\alpha_{ij}} \Delta t^n \leq \delta_0.$$

Then, we have that

$$|\, u_h^n \,| \leq |\, u_0 \,|_{TV(0,1)},$$

where u_h^n is given by an RKDG scheme.

The RKDG methods.

Non-linear hyperbolic problems.

- Positivity-preserving RKDG methods (Shu et al.).
- How to avoid the use of slope limiters?
- Rigorous error analysis for shocks?

We provide a "static condensation" characterization of the solution of the following second-order elliptic model problem:

$$\begin{aligned} \mathrm{c}\,\boldsymbol{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{q} &= f && \text{in } \Omega, \\ \widehat{u} &= u_D && \text{on } \partial\Omega. \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

The HDG methods for diffusion

Static condensation of the exact solution: Local problems and transmission conditions.

We have that the exact solution satisfies the local problems

$$\begin{aligned} \mathrm{c}\,\boldsymbol{q} + \nabla u &= 0 \quad \text{in } K,\\ \nabla \cdot \boldsymbol{q} &= f \quad \text{in } K, \end{aligned}$$

the transmission conditions

$$\begin{aligned} [\![\widehat{u}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o,\\ [\![\widehat{\boldsymbol{q}}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \end{aligned}$$

and the Dirichlet boundary condition

$$\widehat{u} = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

The HDG methods for diffusion

Static condensation of the exact solution: Rewriting the equations.

We can obtain $(\boldsymbol{q}, u)$ in K in terms of $\widehat{u}$ on ∂K and f by solving

$$
\begin{aligned}
\mathrm{c}\,\boldsymbol{q} + \nabla u &= 0 && \text{in } K,\\
\nabla \cdot \boldsymbol{q} &= f && \text{in } K,\\
u &= \widehat{u} && \text{on } \partial K.
\end{aligned}
$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$
\begin{aligned}
[\![\widehat{\boldsymbol{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\
\widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial,
\end{aligned}
$$

where $\widehat{\boldsymbol{q}}$ is the trace of $\boldsymbol{q} = \boldsymbol{q}(\widehat{u}, f)$ on ∂K.

We have that $(\boldsymbol{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathsf{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where

$$\begin{aligned} \mathrm{c}\,\mathbf{Q}_{\widehat{u}} + \nabla \mathsf{U}_{\widehat{u}} &= 0 && \text{in } K, \qquad & \mathrm{c}\,\mathbf{Q}_f + \nabla \mathsf{U}_f &= 0 && \text{in } K,\\ \nabla \cdot \mathbf{Q}_{\widehat{u}} &= 0 && \text{in } K, & \nabla \cdot \mathbf{Q}_f &= f && \text{in } K,\\ \mathsf{U}_{\widehat{u}} &= \widehat{u} && \text{on } \partial K, & \mathsf{U}_f &= 0 && \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} -[\![\widehat{\mathbf{Q}}_{\widehat{u}}]\!] &= [\![\widehat{\mathbf{Q}}_f]\!] && \text{if } F \in \mathcal{E}_h^o,\\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial. \end{aligned}$$

The HDG methods for diffusion

Static condensation of the exact solution. The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\boldsymbol{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where

$$
\begin{aligned}
\mathbf{Q}_{\widehat{u}} + \frac{d}{dx}\mathrm{U}_{\widehat{u}} &= 0 && \text{in } (x_{i-1}, x_i), &\qquad \mathbf{Q}_f + \frac{d}{dx}\mathrm{U}_f &= 0 && \text{in } (x_{i-1}, x_i),\\
\frac{d}{dx}\mathbf{Q}_{\widehat{u}} &= 0 && \text{in } (x_{i-1}, x_i), & \frac{d}{dx}\mathbf{Q}_f &= f && \text{in } (x_{i-1}, x_i),\\
\mathrm{U}_{\widehat{u}} &= \widehat{u} && \text{on } \{x_{i-1}, x_i\}, & \mathrm{U}_f &= 0 && \text{on } \{x_{i-1}, x_i\}.
\end{aligned}
$$

The function $\widehat{u}$ is the solution of

$$
\begin{aligned}
\widehat{\mathbf{Q}}_{\widehat{u}}(x_i^+) - \widehat{\mathbf{Q}}_{\widehat{u}}(x_i^-) &= -\widehat{\mathbf{Q}}_f(x_i^+) + \widehat{\mathbf{Q}}_f(x_i^-) &\quad& \text{for } i = 1, \ldots, I-1,\\
\widehat{u}(x_i) &= u_D(x_i) && \text{for } i = 0, I.
\end{aligned}
$$

The HDG methods for diffusion

Static condensation of the exact solution. The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \dots, I$, with $\mathrm{c} = 1$.

We have that $(\boldsymbol{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where, for $x \in (x_{i-1}, x_i)$,

$$\mathbf{Q}_{\widehat{u}}(x) = -\frac{1}{h}(\widehat{u}_i - \widehat{u}_{i-1}), \qquad \mathbf{Q}_f(x) = -\int_{x_{i-1}}^{x_i} G_x^i(x,s) f(s)\, ds,$$

$$\mathrm{U}_{\widehat{u}}(x) = \varphi_i(x)\, \widehat{u}_i + \varphi_{i-1}(x)\, \widehat{u}_{i-1} \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} G^i(x,s) f(s)\, ds.$$

The function $\widehat{u}$ is the solution of

$$-\frac{1}{h}(\widehat{u}_{i-1} - 2\,\widehat{u}_i + \widehat{u}_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(s)\, f(s)\, ds \qquad \text{for } i = 1, \dots, I-1,$$

$$\widehat{u}(x_i) = u_D(x_i) \qquad \text{for } i = 0, I.$$

The HDG methods for diffusion

Static condensation of the continuous Galerkin method. (Guyan 65)

The continuous Galerkin method provides an approximation to u, $u_h \in W_h(u_D)$, determined by

$$(\mathrm{a}\,\nabla u_h, \nabla w)_\Omega = (f, w)_\Omega \quad \forall w \in W_h(0).$$

where

$$\begin{aligned} W_h &= \{w \in \mathcal{C}^0(\Omega) : \; w|_K \in W(K) \; \forall K \in \Omega_h\}, \\ W_h(g) &= \{w \in W_h : \; w = I_h(g) \text{ on } \partial\Omega\}. \end{aligned}$$

For each element $K \in \Omega_h$,

$$\begin{aligned} W(K) &= W_0(K) \oplus W_\partial(K), \\ W_0(K) &:= \{w \in W(K) : \ w|_{\partial K} = 0\}, \\ W_\partial(K) &:= \{w \in W(K) : \ w|_{\partial K} = 0 \implies w|_K = 0\}. \end{aligned}$$

This implies

$$\begin{aligned} W_h &= W_{0,h} \oplus W_{\mathcal{E}_h} \\ W_{0,h} &:= \{w \in W_h : \ w|_K \in W_0(K) \ \forall K \in \Omega_h\}, \\ W_{\mathcal{E}_h} &:= \{w \in W_h : \ w|_K \in W_\partial(K) \ \forall K \in \Omega_h\}, \end{aligned}$$

and

$$\begin{aligned} M_h &:= \{w|_{\mathcal{E}_h} : \ w \in W_h\}, \\ M_h(g) &:= \{\mu \in M_h : \ \mu|_{\partial\Omega} = I_h(g)\}. \end{aligned}$$

The HDG methods for diffusion

Static condensation of the continuous Galerkin method. Local problems and transmission condition.

We obtain $\mathrm{U} \in W(K)$ in terms of $\widehat{u}_h$ and f by solving

$$\begin{aligned} (\mathrm{a}\,\nabla \mathrm{U}, \nabla w)_K &= (f, w)_K \quad && \forall w \in W_0(K), \\ \mathrm{U} &= \widehat{u}_h && \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}_h \in M_h$ is determined as the solution of

$$\begin{aligned} (\mathrm{a}\,\nabla \mathrm{U}, \nabla w)_\Omega &= (f, w)_\Omega \quad && \forall w \in W_{\mathcal{E}_h}(0), \\ \widehat{u}_h &= I_h(u_D) && \text{on } \partial\Omega. \end{aligned}$$

Note that we have a transmission condition:

$$0 = \langle \mathrm{a}\,\nabla \mathrm{U} \cdot \boldsymbol{n}, \widehat{w} \rangle_{\partial\Omega_h} - (\nabla \cdot (\mathrm{a}\,\nabla \mathrm{U}) + f, w)_{\Omega_h} = \langle \mathrm{a}\,\nabla \mathrm{U} \cdot \boldsymbol{n} + r_\partial, \widehat{w} \rangle_{\partial\Omega_h}$$

We have that $u_h = \mathsf{U}_{\widehat{u}_h} + \mathsf{U}_f$, where

$$
\begin{aligned}
(\mathrm{a}\,\nabla \mathsf{U}_{\widehat{u}_h}, \nabla w)_K &= 0 && \forall w \in W_0(K),\\
\mathsf{U}_{\widehat{u}_h} &= \widehat{u}_h && \text{on } \partial K,\\
(\mathrm{a}\,\nabla \mathsf{U}_f, \nabla w)_K &= (f, w)_K && \forall w \in W_0(K),\\
\mathsf{U}_f &= 0 && \text{on } \partial K,
\end{aligned}
$$

and $\widehat{u}_h$ is the element of $M_h(u_D)$ that solves the global problem

$$
(\mathrm{a}\,\nabla \mathsf{U}_{\widehat{u}_h}, \nabla \mathsf{U}_\mu)_\Omega = (f, \mathsf{U}_\mu)_\Omega \qquad \forall\, \mu \in M_h(0).
$$

The HDG methods for diffusion

Static condensation of the CG method: The original one (Guyan 65)!

The system of equations is

$$K\,[u_h] = [f],$$

and, after splitting the degrees of freedom, it is

$$\begin{bmatrix} K_{00} & K_{0\partial} \\ K_{\partial 0} & K_{\partial\partial} \end{bmatrix} \begin{bmatrix} [\mathsf{U}] \\ [\widehat{u}_h] \end{bmatrix} = \begin{bmatrix} f_0 \\ f_\partial \end{bmatrix}.$$

The solution of the local problems is

$$[\mathsf{U}] = -K_{00}^{-1} K_{0\partial} [\widehat{u}_h] + K_{00}^{-1} [f_0].$$

and the transmission condition

$$(-K_{\partial 0}\, K_{00}^{-1}\, K_{0\partial} + K_{\partial\partial})[\widehat{u}_h] = -K_{\partial 0}\, K_{00}^{-1} [f_0] + [f_\partial].$$

For $W(K) := \mathcal{P}_k(K)$, the solution of the local problems are

$$\mathrm{U}_{\widehat{u}}(x) = \varphi_i(x)\,\widehat{u}_i + \varphi_{i-1}(x)\,\widehat{u}_{i-1} \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} G_h^i(x,s) f(s)\,ds,$$

and where the global problem for the values $\{\widehat{u}_i\}_{i=0}^N$ is

$$-\frac{1}{h}(\widehat{u}_{i-1} - 2\,\widehat{u}_i + \widehat{u}_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(s)\, f(s)\,ds \quad \text{for } i = 1, \dots, N-1,$$

$$\widehat{u}_j = u_D(x_j) \quad \text{for } j = 0, N.$$

The HDG methods for diffusion

Static condensation of mixed methods (deVeubeke 65).

The function $(\boldsymbol{q}_h, u_h)$ is the only element of $\boldsymbol{\mathcal{V}}_h \times W_h$ satisfying the equations

$$\begin{aligned}(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_\Omega - (u_h, \nabla\cdot\boldsymbol{v})_\Omega &= -\langle u_D, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega} && \forall \boldsymbol{v}\in\boldsymbol{\mathcal{V}}_h,\\ (\nabla\cdot\boldsymbol{q}_h, w)_\Omega &= (f, w)_\Omega && \forall w\in W_h.\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{V}}_h &= \{\boldsymbol{v}\in\boldsymbol{H}(div,\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}(K) \ \ \forall K\in\Omega_h\}.\\ W_h &= \{w\in L^2(\Omega) : \quad w|_K \in W(K)\ \forall K\in\Omega_h\}.\end{aligned}$$

We define $(\mathbf{Q}, \mathsf{U}) \in \boldsymbol{V}(K) \times W(K)$ in terms of $\widehat{u}_h$ and f as the solution of the local problem

$$\begin{aligned}
(\mathrm{c}\,\mathbf{Q}, \boldsymbol{v})_K - (\mathsf{U}, \nabla\cdot\boldsymbol{v})_K &= \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
(\nabla\cdot\mathbf{Q}, w)_K &= (f, w)_K && \forall w \in W(K).
\end{aligned}$$

The function $\widehat{u}_h$ in the space M_h is such that

$$\begin{aligned}
[\![\mathbf{Q}]\!] &= 0 && \text{on } \mathcal{E}_h^o,\\
\widehat{u}_h &= u_D && \text{on } \partial\Omega.
\end{aligned}$$

The weak form of the transmission condition is

$$\langle \mathbf{Q}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} = \langle \mathbf{Q}, \mu\rangle_{\partial\Omega_h\setminus\partial\Omega} = \langle\, [\![\mathbf{Q}]\!], \mu\rangle_{\mathcal{E}_h^o} = 0 \qquad \forall \mu \in M_h(0).$$

The HDG methods for diffusion

Static condensation of mixed methods: A characterization of the approximate solution.

We have that $(\boldsymbol{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where, $\forall K \in \Omega_h$,

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{Q}_\mu, \boldsymbol{v})_K - (\mathsf{U}_\mu, \nabla\cdot\boldsymbol{v})_K &= -\langle \mu, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} && \forall\, \boldsymbol{v} \in \boldsymbol{V}(K),\\
(\nabla\cdot\mathbf{Q}_\mu, w)_K &= 0 && \forall\, w \in W(K),\\
(\mathrm{c}\,\mathbf{Q}_f, \boldsymbol{v})_K - (\mathsf{U}_f, \nabla\cdot\boldsymbol{v})_K &= 0 && \forall\, \boldsymbol{v} \in \boldsymbol{V}(K),\\
(\nabla\cdot\mathbf{Q}_f, w)_K &= (f, w)_K && \forall\, w \in W(K),
\end{aligned}
$$

and the function $\widehat{u}_h$ is the element of $M_h(u_D)$ which solves the global problem

$$
(\mathrm{c}\,\mathbf{Q}_{\widehat{u}_h}, \mathbf{Q}_\mu)_{\Omega_h} = (f, \mathsf{U}_\mu)_{\Omega_h} \quad \forall\, \mu \in M_h(0).
$$

Note that

$$
0 = \langle \mathbf{Q}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} = \langle \mathbf{Q}_{\widehat{u}_h}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} + \langle \mathbf{Q}_f\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} = -(\mathrm{c}\,\mathbf{Q}_{\widehat{u}_h}, \mathbf{Q}_\mu)_{\Omega_h} + (\mathsf{U}_\mu, f)_{\partial\Omega_h}.
$$

The HDG methods for diffusion

Static condensation of mixed methods: The original hybridization (deVeubeqe 65)!

The system of equations is

$$\begin{bmatrix} \mathcal{A} & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} [\boldsymbol{q}_h] \\ [u_h] \end{bmatrix} = \begin{bmatrix} [u_D] \\ [f] \end{bmatrix}.$$

which, after hybridization, becomes

$$\begin{bmatrix} A & B & C \\ B^t & 0 & 0 \\ C^t & 0 & 0 \end{bmatrix} \begin{bmatrix} [\mathbf{Q}] \\ [\mathsf{U}] \\ [\widehat{u}_h] \end{bmatrix} = \begin{bmatrix} -C_\partial[u_D] \\ [f] \\ 0 \end{bmatrix}.$$

The solution of the local problems is

$$\begin{bmatrix} [\mathbf{Q}] \\ [\mathsf{U}] \end{bmatrix} = \begin{bmatrix} A & B \\ B^t & 0 \end{bmatrix}^{-1} \begin{bmatrix} -C[\widehat{u}_h] - C_\partial[u_D] \\ [f] \end{bmatrix}.$$

and the transmission condition is $H[\,\widehat{u}_h\,] = H_\partial[\,u_D\,] + J\,[\,f\,]$,

$$H := C^t\,(A^{-1} - A^{-1}B\,(B^tA^{-1}B)^{-1}\,B^tA^{-1})\,C.$$

For $\boldsymbol{V}(K) \times W(K) := \mathcal{P}_{k+1}(K) \times \mathcal{P}_k(K)$, the solution of the local problems is

$$\mathbf{Q}_{\widehat{u}}(x) = -\frac{\widehat{u}_i - \widehat{u}_{i-1}}{h}, \qquad \mathbf{Q}_f(x) = \int_{x_{i-1}}^{x_i} H_h^i(x,s) f(s)\, ds,$$

$$\mathrm{U}_{\widehat{u}}(x) = \varphi_i(x)\, \widehat{u}_i + \varphi_{i-1}(x)\, \widehat{u}_{i-1}, \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} G_h^i(x,s) f(s)\, ds,$$

and the global problem for the values $\{\widehat{u}_i\}_{i=0}^N$ is

$$-\frac{1}{h}(\widehat{u}_{i-1} - 2\, \widehat{u}_i + \widehat{u}_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(s)\, f(s)\, ds \quad \text{for } i = 1, \dots, N-1,$$

$$\widehat{u}_i = u_D(x_j) \quad \text{for } i = 0, N.$$

- The HDG methods are obtained by constructing discrete versions (based on discontinuous Galerkin methods) of the above characterization of the exact solution.
- In this way, the globally coupled degrees of freedom will be those of the corresponding global formulations.

On the element $K \in \Omega_h$, we define $(\boldsymbol{q}_h, u_h)$ terms of $(\widehat{u}_h, f)$ as the element of $\boldsymbol{V}(K) \times W(K)$ such that

$$
\begin{aligned}
(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_K - (u_h, \nabla\cdot\boldsymbol{v})_K + \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\
-(\boldsymbol{q}_h, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K) \times W(K)$, where

$$
\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial K.
$$

For each face $F \in \mathcal{E}_h^o$, we take $\widehat{u}_h|_F$ in the space $M(F)$. We determine $\widehat{u}_h$ by requiring that,

$$\langle \mu, [\![\widehat{\boldsymbol{q}}_h]\!] \rangle_F = 0 \quad \forall\, \mu \in M(F) \quad \text{if } F \in \mathcal{E}_h^o,$$
$$\widehat{u}_h = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

All the HDG methods are generated by choosing the local spaces $\boldsymbol{V}(K)$, $W(K)$, $M(F)$ and the stabilization function τ.

The approximate solution $(\boldsymbol{q}_h, u_h, \widehat{u}_h)$ is the element of the space $\boldsymbol{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$
\begin{aligned}
(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,
\end{aligned}
$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h(0)$, where

$$
\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.
$$

Suppose that the transmission condition implies that $[\![\widehat{\boldsymbol{q}}_h]\!] = 0$ on a face $F \in \mathcal{E}_h^o$. Then, on that face, we have that

$$[\![\boldsymbol{q}_h]\!] + \tau^+({u_h}^+ - \widehat{u}_h) + \tau^-({u_h}^- - \widehat{u}_h) = 0,$$

which holds if

$$\widehat{u}_h = \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-} [\![\boldsymbol{q}_h]\!],$$

$$\widehat{\boldsymbol{q}}_h = \frac{\tau^- {\boldsymbol{q}_h}^+ + \tau^+ {\boldsymbol{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+ \tau^-}{\tau^+ + \tau^-} [\![u_h]\!]$$

provided $\tau^+ + \tau^- > 0$.

Formulation for $(u_h, \widehat{u}_h)$

Characterization of the approximate solution (D.Arnold and F.Brezzi, RAIRO, 1985; ABCD, SINUM, 02; B.C. and K.Shi, C&F, 2014.)

For any $(w, \mu) \in W_h \times M_h$, define $\boldsymbol{q}_{w,\mu} \in \boldsymbol{V}_h$ as the solution of

$$(\mathrm{c}\, \boldsymbol{q}_{w,\mu}, \boldsymbol{v})_{\Omega_h} - (w, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \mu, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} = 0,$$

for all $\boldsymbol{v} \in V_h$.

The approximate solution is $(\boldsymbol{q}_{u_h,\widehat{u}_h}, u_h, \widehat{u}_h)$ where $(u_h, \widehat{u}_h)$ is the element of $W_h \times M_h(u_D)$ satisfying the equations

$$(\nabla \cdot \boldsymbol{q}_{u_h,\widehat{u}_h}, w)_{\Omega_h} + \langle \tau(u_h - \widehat{u}_h), w \rangle_{\partial\Omega_h} = (f, w)_{\Omega_h},$$
$$\langle \mu, \boldsymbol{q}_{u_h,\widehat{u}_h} \cdot \boldsymbol{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial\Omega_h} = 0,$$

for all $(w, \mu) \in W_h \times M_h(0)$.

Formulation for $(u_h, \widehat{u}_h)$

Characterization of the approximate solution (D.Arnold and F.Brezzi, RAIRO, 1985; ABCD, SINUM, 02; B.C. and K.Shi, C&F, 2014.)

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The approximate solution is $(\boldsymbol{q}_{u_h,\widehat{u}_h}, u_h, \widehat{u}_h)$ where $(u_h, \widehat{u}_h)$ is the element of $W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned}(\mathrm{c}\, \boldsymbol{q}_{u_h,\widehat{u}_h}, \boldsymbol{q}_{w,\mu})_{\Omega_h} + \langle \mu, \boldsymbol{q}_{u_h,\widehat{u}_h} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} + \langle \tau(u_h - \widehat{u}_h), w \rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h}, \\ \langle \mu, \boldsymbol{q}_{u_h,\widehat{u}_h} \cdot \boldsymbol{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial\Omega_h} &= 0,\end{aligned}$$

for all $(w, \mu) \in W_h \times M_h(0)$.

Formulation for $(u_h, \widehat{u}_h)$

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for all $\boldsymbol{v} \in V_h$.
The approximate solution is $(\boldsymbol{q}_{u_h,\widehat{u}_h}, u_h, \widehat{u}_h)$ where $(u_h, \widehat{u}_h)$ is the element of $W_h \times M_h(u_D)$ satisfying the equations

$$(\mathrm{c}\,\boldsymbol{q}_{u_h,\widehat{u}_h}, \boldsymbol{q}_{w,\mu})_{\Omega_h} + \langle \tau(u_h - \widehat{u}_h), w - \mu\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h},$$

for all $(w, \mu) \in W_h \times M_h(0)$.

Formulation for $(u_h, \widehat{u}_h)$

The associated minimization property. (H. Kabbaria, A. Lew, and B.C., 14; B.C. and K.Shi, 14; B.C. and J.Shen, 15)

The function $(u_h, \widehat{u}_h)$ minimizes the quadratic functional

$$J_h(w,\mu) := \frac{1}{2}(\mathrm{c}\,\boldsymbol{q}_{w,\mu}, \boldsymbol{q}_{w,\mu})_{\Omega_h} + \frac{1}{2}\langle \tau(w-\mu), (w-\mu)\rangle_{\partial\Omega_h} - (f, w)_{\Omega_h},$$

over the functions $(w,\mu) \in W_h \times M_h(u_D)$.

Formulation for $\widehat{u}_h$

Characterization of the approximate solution (B.C. and J.Gopalakrishnan, SINUM, 2005; B.C. and J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

We have that $(\boldsymbol{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathrm{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathrm{U}_f)$ where

$$(\mathbf{Q}_{\widehat{u}_h}, \mathrm{U}_{\widehat{u}_h}) := (\mathbf{Q}(\widehat{u}_h, 0), \mathrm{U}(\widehat{u}_h, 0)), \quad (\mathbf{Q}_f, \mathrm{U}_f) := (\mathbf{Q}(0, f), \mathrm{U}(0, f)).$$

where $(\mathbf{Q}(\widehat{u}_h, f), \mathrm{U}(\widehat{u}_h, f))$ is the linear mapping that associates $(\widehat{u}_h, f)$ to $(\boldsymbol{q}_h, u_h)$, and where the numerical trace $\widehat{u}_h$ is the element of the space

$$M_h(u_D) := \{\mu \in L^2(\mathcal{E}_h) : \quad \mu|_F \in M(F) \ \forall\, F \in \mathcal{E}_h, \quad u_h|_{\partial\Omega} := P_\partial u_D\},$$

satisfying the equations

$$a_h(\widehat{u}_h, \mu) = \ell_h(\mu) \quad \forall\, \mu \in M_h(0),$$

where $a_h(\mu, \lambda) := -\langle \mu, \widehat{\mathbf{Q}}_\lambda \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}$, and $\ell_h(\mu) := \langle \mu, \widehat{\mathbf{Q}}_f \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}$.

Formulation for $\widehat{u}_h$

The associated minimization problem (B.C. and K.Shi, C&F, 14; B.C. and J.Shen, 15)

Theorem

We have that

$$a_h(\mu,\lambda) = (\mathrm{c}\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_{\partial\Omega_h} + \langle \tau(\mathsf{U}_\mu - \mu), (\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h},$$
$$\ell_h(\mu) = (f, \mathsf{U}_\mu)_{\partial\Omega_h}.$$

Moreover, $a_h(\cdot,\cdot)$ *is positive definite on* $M_h(0)\times M_h(0)$.

The numerical trace $\widehat{u}_h$ minimizes the quadratic functional

$$J_h(\eta) := \frac{1}{2} a_h(\eta,\eta) - \ell_h(\eta),$$

over the functions η in $M_h(u_D)$.

Theorem

If $\boldsymbol{V}(K) = \boldsymbol{\mathcal{P}}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $M(F) = \mathcal{P}_k(K)$, $k \geq 0$,the condition number of $a_h(\cdot, \cdot)$ (on $M_{h,0} \times M_{h,0}$) is of order

$$(1 + (\tau^*\, h)^2) h^{-2}.$$

Here $\tau^* := \max_{K \in \Omega_h} \tau|_{\partial K \setminus F_K^*}$, where F_K^* is an arbitrary face of the simplex K.

Note that the matrix is invertible even if $\tau \equiv 0$!

Existence and uniqueness.

The local problems are well defined.

Theorem

The local solver on K is well defined if

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \boldsymbol{V}(K)$.

The system is square. Set $\widehat{u}_h = 0$ and $f = 0$.
For $(\boldsymbol{v}, w) := (\boldsymbol{q}_h, u_h)$, the equations read

$$\begin{aligned}(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{q}_h)_K - (u_h, \nabla\cdot\boldsymbol{q}_h)_K &= 0,\\ -(\boldsymbol{q}_h, \nabla u_h)_K + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, u_h\rangle_{\partial K} &= 0.\end{aligned}$$

Hence

$$(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{q}_h)_K + \langle (\widehat{\boldsymbol{q}}_h - \boldsymbol{q}_h)\cdot\boldsymbol{n}, u_h\rangle_{\partial K} = 0,$$

and since $\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau(u_h)$, we get

$$(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{q}_h)_K + \langle \tau\,(u_h), u_h\rangle_{\partial K} = 0.$$

This implies that $\boldsymbol{q}_h = 0$ on K, and that $u_h = 0$ on ∂K.

Existence and uniqueness.

Proof.

Now, the first equation defining the local problems reads

$$-(u_h, \nabla \cdot \boldsymbol{v})_K = 0,$$

for all $\boldsymbol{v} \in \boldsymbol{V}(K)$. Hence

$$(\nabla u_h, \boldsymbol{v})_K = 0,$$

and so $\nabla u_h = 0$. This proves the result.

Existence and uniqueness.

The numerical trace $\widehat{u}_h$ is well defined.

Theorem

The numerical trace $\widehat{u}_h$ is well defined if, for each $K \in \partial\Omega_h$,

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \boldsymbol{V}(K)$.

Existence and uniqueness.

Proof.

The system is square. Set $u_D = 0$ and $f = 0$. For $\mu := \widehat{u}_h$, the equation reads

$$0 = \sum_{F \in \mathcal{E}_h^o} \langle \widehat{u}_h, [\![\widehat{\boldsymbol{q}}_h]\!] \rangle_F = \sum_{K \in \Omega_h} \langle \widehat{u}_h, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \rangle_{\partial K} =: \langle \widehat{u}_h, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \rangle_{\partial \Omega_h}.$$

Note that

$$\begin{aligned}
-\langle \widehat{u}_h, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} &= - \langle \widehat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau (u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} - \langle u_h, \tau (u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau (u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} - \langle u_h, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} + \langle u_h, \boldsymbol{q}_h \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau (u_h - \widehat{u}_h) \rangle_{\partial \Omega_h}
\end{aligned}$$

Existence and uniqueness.

Proof.

For $(\boldsymbol{v}, w) := (\boldsymbol{q}_h, u_h)$, the equations of the local problems read

$$\begin{aligned} (\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{q}_h)_K - (u_h, \nabla\cdot\boldsymbol{q}_h)_K + \langle \widehat{u}_h, \boldsymbol{q}_h\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\ -(\boldsymbol{q}_h, \nabla u_h)_K + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, u_h\rangle_{\partial K} &= 0. \end{aligned}$$

Then

$$-\langle \widehat{u}_h, \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = (\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{q}_h)_{\Omega_h} + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h)\rangle_{\partial\Omega_h}.$$

As a consequence, $\langle \widehat{u}_h, \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0$ implies $\boldsymbol{q}_h = 0$ on Ω_h and $u_h = \widehat{u}_h$ on $\partial\Omega_h$.

Existence and uniqueness.

Proof.

Now, the first equation defining the local problems reads

$$-(u_h, \nabla \cdot \boldsymbol{v})_K + \langle u_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0,$$

for all $\boldsymbol{v} \in \boldsymbol{V}(K)$. Hence

$$(\nabla u_h, \boldsymbol{v})_K = 0,$$

and so $\nabla u_h = 0$.

This shows that u_h is a constant and, since $u_h = \widehat{u}_h = 0$ on $\partial\Omega$, we can conclude that $u_h = 0$ on Ω_h. We now have that $\widehat{u}_h = u_h = 0$ on $\partial\Omega_h$. This proves the result.

We seek HDG methods for which the local averages of the error $u - u_h$, converge faster than the errors $u - u_h$ and $\boldsymbol{q} - \boldsymbol{q}_h$.

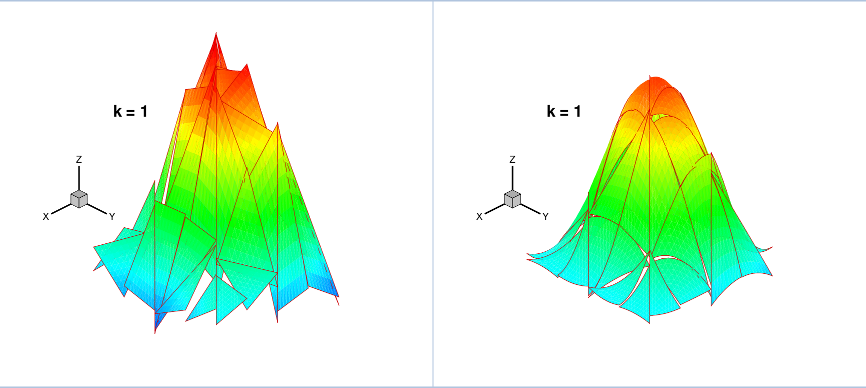
If this property holds, we introduce a new approximation $u_h^\star$. On each element K it lies in the space $W^*(K)$ and defined by

$$
\begin{aligned}
(\nabla {u_h}^\star, \nabla w)_K &= -(\mathrm{c}\boldsymbol{q}_h, \nabla w)_K \qquad \text{for all } w \in W^*(K),\\
({u_h}^\star, 1)_K &= (u_h, 1)_K,
\end{aligned}
$$

Then $u - u_h^\star$ will converge faster than $u - u_h$. This does happen for mixed methods!

Illustration of the postprocessing.

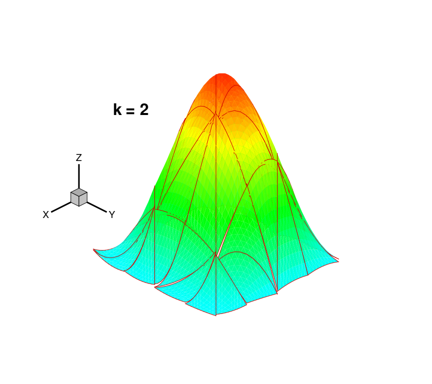
An HDG method for linear elasticity.(S.-C. Soon, B.C. and H. Stolarski, 2008.)

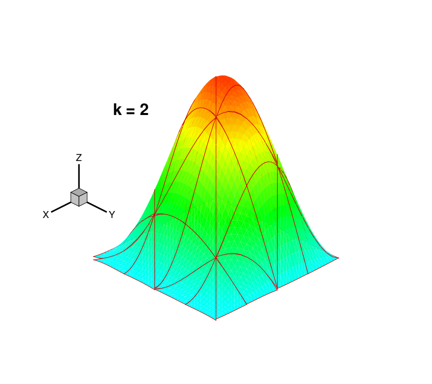


Comparison between the approximate solution (left) and the post-processed solution (right) for linear polynomial approximations.

Illustration of the postprocessing.

An HDG method for linear elasticity.(S.-C. Soon, B.C. and H. Stolarski, 2008.)

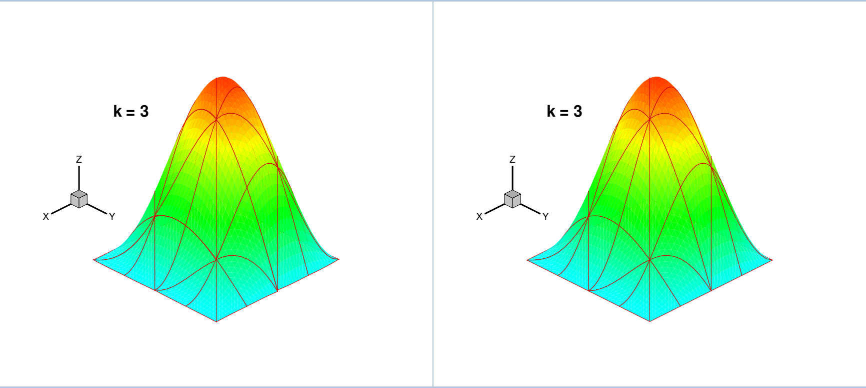




Comparison between the approximate solution (left) and the post-processed solution (right) for quadratic polynomial approximations.

Illustration of the postprocessing.

An HDG method for linear elasticity.(S.-C. Soon, B.C. and H. Stolarski, 2008.)



Comparison between the approximate solution (left) and the post-processed solution (right) for cubic polynomial approximations.

First superconvergent HDG methods. (B.C, B.Dong and J.Guzman, 08; B.C., J.Gopalakrishnan and F.-J. Sayas, 10)

The first superconvergent HDG method: the SFH method

Method	τ	$\boldsymbol{q}_h$	u_h	$\overline{u}_h$	k
RT	0	$k+1$	$k+1$	$k+2$	≥ 0
SFH	> 0	$k+1$	$k+1$	$k+2$	≥ 1
LDG-H	$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$	≥ 1
BDM	0	$k+1$	k	$k+2$	≥ 2

Sufficient conditions for superconvergence

The conditions on the local spaces.(B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

Theorem

Suppose that the local spaces are such that

$$
\begin{aligned}
& \boldsymbol{V}(K)\cdot\boldsymbol{n} + W(K) && \subset M(\partial K),\\
\mathcal{P}_0(K)\times\mathcal{P}_0(K) \subset\ & \nabla W(K)\times\nabla\cdot\boldsymbol{V}(K) && \subset \widetilde{\boldsymbol{V}}(K)\times\widetilde{W}(K),\\
& \widetilde{\boldsymbol{V}}^{\perp}\cdot\boldsymbol{n}\oplus\widetilde{W}^{\perp} && = M(\partial K).
\end{aligned}
$$

Then there is a stabilization function τ such that the HDG method superconverges.

Sufficient conditions for superconvergence.

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

method	$\boldsymbol{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k}(K) \times P^{k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}_3^k(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}^{Q}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}_2^k(K)$	$Q^k(K)$

Sufficient conditions for superconvergence.

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a cube. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

method	$\boldsymbol{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k,k}(K) \times P^{k,k+1,k}(K) \times P^{k,k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}^k_7(K)$	$Q^k(K)$
$\mathbf{HDG}^Q_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}^k_6(K)$	$Q^k(K)$

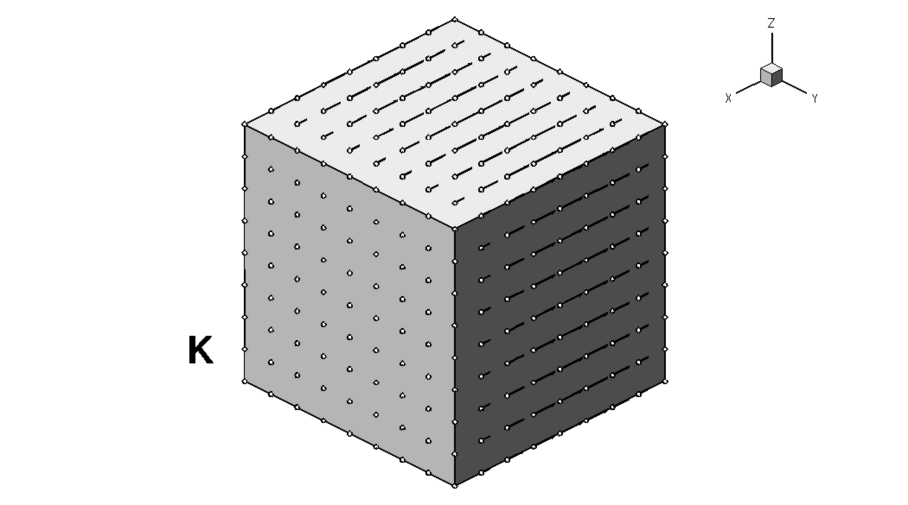
Sufficient conditions for superconvergence.

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square or a cube. (B.C., W.Qiu and K.Shi, Math. Comp.,2012 + SINUM, 2012.)

method	τ	$\|\boldsymbol{q} - \boldsymbol{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u - u_h^\star\|_\Omega$
$\mathbf{RT}_{[k+1]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{TNT}_{[k]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}^{Q}_{[k]}$	$\mathcal{O}(1) > 0$	$k+1$	$k+2$	$k+2$

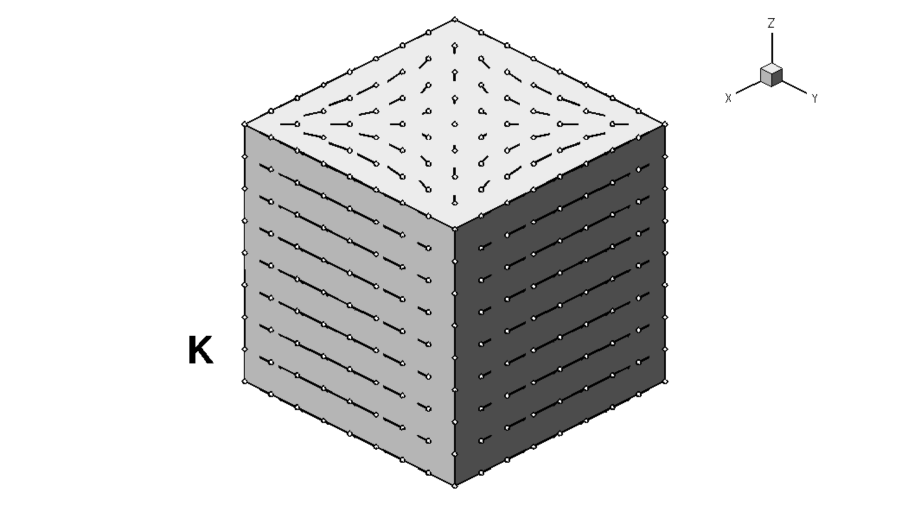
Sufficient conditions for superconvergence.

TNT in 3D: The space $\boldsymbol{H}_7^k(K)$.



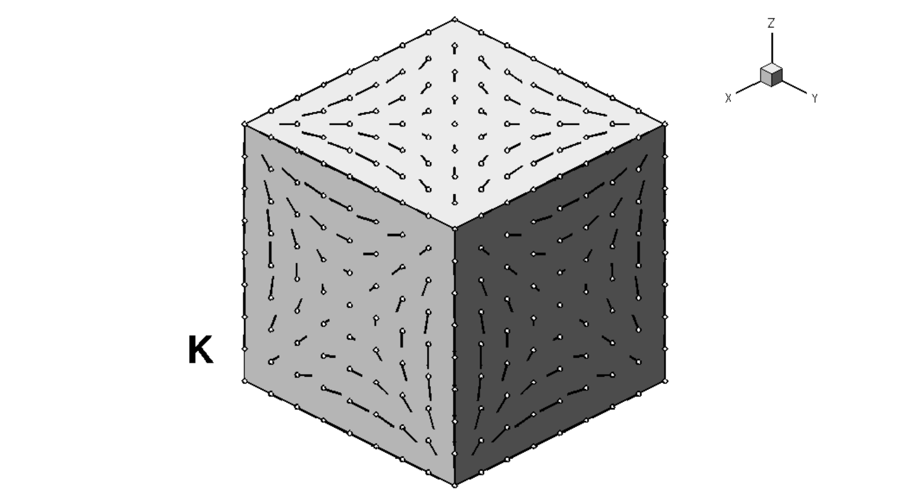
Sufficient conditions for superconvergence.

TNT in 3D: The space $\boldsymbol{H}_7^k(K)$.



Sufficient conditions for superconvergence.

TNT in 3D: The space $\boldsymbol{H}_7^k(K)$.



The theory of M-decompositions.

(B.C., G.Fu, F.-J. Sayas, Math. Comp., to appear; B.C. and G.Fu, 2D+3D, M^2AN, to appear)

Definition (The M-decomposition)

We say that $\boldsymbol{V} \times W$ admits an M-decomposition when

(a) $\mathrm{tr}(\boldsymbol{V} \times W) \subset M$,

and there exists a subspace $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ of $\boldsymbol{V} \times W$ satisfying

(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \widetilde{\boldsymbol{V}} \times \widetilde{W}$,

(c) $\mathrm{tr} : \widetilde{\boldsymbol{V}}^{\perp} \times \widetilde{W}^{\perp} \to M$ is an isomorphism.

Here $\widetilde{\boldsymbol{V}}^{\perp}$ and $\widetilde{W}^{\perp}$ are the $L^2(K)$-orthogonal complements of $\widetilde{\boldsymbol{V}}$ in $\boldsymbol{V}$, and of $\widetilde{W}$ in W, respectively.

The theory of M-decompositions.

A characterization of M-decompositions. (B.C., G.Fu, F.-J. Sayas, Math. Comp., to appear)

$$I_M(\boldsymbol{V} \times W) := \dim M - \dim\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} - \dim\{w|_{\partial K} : w \in W, \nabla w = 0\}.$$

Theorem

For a given space of traces M, the space $\boldsymbol{V} \times W$ admits an M-decomposition if and only if

(a) $\mathrm{tr}(\boldsymbol{V} \times W) \subset M$,

(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \boldsymbol{V} \times W$,

(c) $I_M(\boldsymbol{V} \times W) = 0$.

In this case, we have

$$M = \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W, \nabla w = 0\},$$

where the sum is orthogonal.

Table: Construction of spaces $\boldsymbol{V} \times W$ admitting an M-decomposition, where the space of traces $M(\partial K)$ includes the constants. The given space $\boldsymbol{V}_g \times W_g$ satisfies the inclusion properties (a) and (b).

$\boldsymbol{V}$	W	$\nabla \cdot \boldsymbol{V}$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}} \oplus \delta \boldsymbol{V}_{\text{fillW}}$	W_g (if $\supset \mathcal{P}_0(K)$)	$= W_g$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}$	W_g (if $\supset \mathcal{P}_0(K)$)	$\subset W_g$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}$	$\nabla \cdot \boldsymbol{V}_g$ (if $\supset \mathcal{P}_0(K)$)	$= \nabla \cdot \boldsymbol{V}_g$

$\delta \boldsymbol{V}$	$\nabla \cdot \delta \boldsymbol{V}$	$\gamma \delta \boldsymbol{V}$	$\dim \delta \boldsymbol{V}$
$\delta \boldsymbol{V}_{\text{fillM}}$	$\{\boldsymbol{0}\}$	$\subset M, \cap \gamma \boldsymbol{V}_{g_s} = \{0\}$	$I_M(\boldsymbol{V}_g \times W_g)$
$\delta \boldsymbol{V}_{\text{fillW}}$	$\subset W_g, \cap \nabla \cdot \boldsymbol{V}_g = \{0\}$	$\subset M$	$I_S(\boldsymbol{V}_g \times W_g)$

Construction of M-decompositions

Theorem

Let $\boldsymbol{V}_g \times W_g$ satisfy properties (a) and (b) of an M-decomposition. Assume that $\delta \boldsymbol{V}_{\text{fillM}}$ satisfies the following hypotheses:

(a) $\nabla \cdot \delta \boldsymbol{V}_{\text{fillM}} = \{0\}$,

(b) $\delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K} \subset M$,

(c) $\delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K}$ *and* $\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \; \boldsymbol{v} \in \boldsymbol{V}, \; \nabla \cdot \boldsymbol{v} = 0\}$ *are linearly independent,*

(d) $\dim \delta \boldsymbol{V}_{\text{fillM}} = \dim \delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K} = I_M(\boldsymbol{V}_g \times W_g)$

Then, $(\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}) \times W_g$ admits an M-decomposition.

A construction of M-decompositions

A three-step procedure to construct the filling space $\delta\boldsymbol{V}_{\text{fillM}}$

(1) Characterize the trace space $\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K}:\ \boldsymbol{v}\in\boldsymbol{V},\ \nabla\cdot\boldsymbol{v}=0\}$

(2) Find a trace space $C_M\subset M(\partial K)$ such that

$$C_M\oplus\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K}:\boldsymbol{v}\in\boldsymbol{V},\nabla\cdot\boldsymbol{v}=0\}=\{\mu\in M:\langle\mu,1\rangle_{\partial K}=0\}$$

note that the dimension of the space C_M is equal to $I_M(\boldsymbol{V}\times W)$

(3) Set $\delta\boldsymbol{V}_{\text{fillM}}:=\{\boldsymbol{v}_\mu:\ \mu\in C_M\}$, where $\boldsymbol{v}_\mu$ is divergence-free function such that $\boldsymbol{v}_\mu\cdot\boldsymbol{n}|_{\partial K}=\mu$

A construction of M-decompositions

The M-indexes for different elements

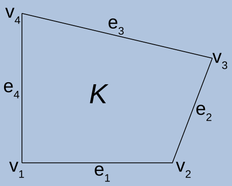
$$\boldsymbol{V} \times W \times M := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K)$$

2D element	$I_M(\boldsymbol{V} \times W)$	3D element	$I_M(\boldsymbol{V} \times W)$
triangle	0 $(k\geq 0)$	tetrahedron	0 $(k\geq 0)$
quadrilateral	1 $(k=0)$, 2 $(k\geq 1)$	pyramid	1 $(k=0)$, 3 $(k\geq 1)$
pentagon	2 $(k=0)$, 4 $(k=1)$, 5 $(k\geq 2)$	prism[1]	1 $(k=0)$, 3 $(k\geq 1)$
hexagon	3 $(k=0)$, 6 $(k=1)$, 8 $(k=2)$, 9 $(k\geq 3)$	hexahedron[2]	2 $(k=0)$, 6 $(k=1)$, 9 $(k\geq 2)$

[1] no parallel faces

A construction of M-decompositions

An example of $\delta \boldsymbol{V}_{\mathrm{fillM}}$ on a quadrilateral



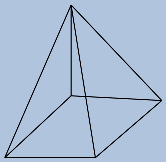
$$\boldsymbol{V} \times W \times M := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K),$$

$$\delta \boldsymbol{V}_{\mathrm{fillM}} := \mathrm{span}\{\nabla\times(\xi_4 \lambda_4^k), \nabla\times(\xi_4 \lambda_3^k)\}.$$

- λ_i is a linear function that vanishes on edge e_i.
- $\xi_4 \in H^1(K)$ is a function such that its trace on each edge is linear and vanishes at the vertices v_1, v_2, and v_3.

A construction of M-decompositions

An example of $\delta \boldsymbol{V}_{\mathrm{fillM}}$ on the reference pyramid



$$K := \{(x, y, z) : 0 < x, 0 < y, 0 < z, x + z < 1, y + z < 1\}$$

$$\boldsymbol{V} \times W \times M := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K)$$

$$\delta \boldsymbol{V}_{\mathrm{fillM}} := \begin{cases} \mathrm{span}\{\nabla \times (\frac{x\,y}{1-z} \nabla z)\} & \text{if } k = 0 \\ \mathrm{span}\{\nabla \times (\frac{x\,y^{k+1}}{1-z} \nabla z), \nabla \times (\frac{y\,x^{k+1}}{1-z} \nabla z), \nabla \times (\frac{x\,y}{1-z} \nabla x)\} & \text{if } k \geq 1 \end{cases}$$

Theorem

Let the space $\boldsymbol{V} \times W$ admit an M-decomposition and assume that $\nabla \cdot \boldsymbol{V}_g \subsetneq W$. Then,

$$\boldsymbol{V} \times \nabla \cdot \boldsymbol{V} \text{ admits an } M\text{-decomposition.}$$

Moreover, let $\delta \boldsymbol{V}_{\mathrm{fillW}}$ satisfy the following hypotheses:

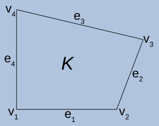
(a) $\delta \boldsymbol{V}_{\mathrm{fillW}} \cdot \boldsymbol{n}|_{\partial K} \subset M$,

(b) $\nabla \cdot \delta \boldsymbol{V}_{\mathrm{fillW}} \oplus \nabla \cdot \boldsymbol{V} = W_g$,

(c) $\dim \delta \boldsymbol{V}_{\mathrm{fillW}} = \dim \nabla \cdot \delta \boldsymbol{V}_{\mathrm{fillW}}$,

Then $(\boldsymbol{V} \oplus \delta \boldsymbol{V}_{\mathrm{fillW}}) \times W$ admits an M-decomposition.

For the above choices of spaces, we can set stabilization operator $\tau = 0$ in and obtain hybridized mixed methods.

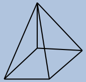


$$\boldsymbol{V}^{hdg} \times W^{hdg} \times M := \boldsymbol{\mathcal{P}}_k(K) \oplus \delta \boldsymbol{V}_{\text{fillM}} \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K),$$

$$\delta \boldsymbol{V}_{\text{fillM}} := \operatorname{span}\{\nabla\times(\xi_4\lambda_4^k), \nabla\times(\xi_4\lambda_3^k)\}.$$

$$\delta \boldsymbol{V}_{\text{fillW}} := \boldsymbol{x}\,\mathcal{P}kK.$$

	$\boldsymbol{V}$	W	M	τ
UMX	$\boldsymbol{V}^{hdg} \oplus \delta \boldsymbol{V}_{\text{fillW}}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	0
HDG	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	>0
LMX	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(\partial K)$	0



$$\boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}} \times M := \boldsymbol{\mathcal{P}}_k(K) \oplus \delta\boldsymbol{V}_{\mathrm{fillM}} \times \mathcal{P}_k(K) \times \mathcal{P}_k(\partial K),\ k \geq 1$$

$$\delta\boldsymbol{V}_{\mathrm{fillM}} := \mathrm{span}\{\nabla\times(\frac{x\,y^{k+1}}{1-z}\nabla z),\ \nabla\times(\frac{y\,x^{k+1}}{1-z}\nabla z),\ \nabla\times(\frac{x\,y}{1-z}\nabla x)\}.$$

$$\delta\boldsymbol{V}_{\mathrm{fillW}} := \boldsymbol{x}\,\mathcal{P}/kK.$$

	$\boldsymbol{V}$	W	M	τ
UMX	$\boldsymbol{V}^{hdg} \oplus \delta\boldsymbol{V}_{\mathrm{fillW}}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	0
HDG	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(\partial K)$	> 0
LMX	$\boldsymbol{V}^{hdg}$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(\partial K)$	0

History of convergence of LDG-H with $k = 1$

h	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate
	$\tau = 1$					
0.1	0.15E-2	-	0.83E-2	-	0.52E-2	-
0.05	0.18E-3	3.06	0.16E-2	2.36	0.10E-2	2.34
0.025	0.23E-4	3.03	0.28E-3	2.52	0.19E-3	2.43
0.0125	0.28E-5	3.02	0.44E-4	2.68	0.35E-4	2.46

History of convergence of M-decompositions with $k = 1$

h	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate	$\|u - u_h^\star\|_{\Omega_h}$	rate
	$\tau = 1$					
0.1	0.15E-2	-	0.26E-2	-	0.17E-2	-
0.05	0.18E-3	3.06	0.31E-3	3.06	0.21E-3	3.02
0.025	0.23E-4	3.03	0.38E-4	3.03	0.27E-4	2.95
0.0125	0.28E-5	3.02	0.47E-5	3.02	0.35E-5	2.96

The theory of M-decompositions

Provides:

1. A systematic way of constructing superconvergent HDG and hybridized mixed methods for elements of arbitrary shapes.
2. A systematic approach to satisfying elementwise inf-sup conditions, stabilized (HDG) or not (mixed methods).
3. A systematic way of constructing finite element commuting diagrams.

The evolution of HDG methods.

Steady-state diffusion

- Relation with old DG methods. (C. Gopalakrishnan, Lazarov, 09; C., Guzman, Wang, 09).
- Relation with mixed methods:
  - The SFH method + relation with SDG method (C., Dong, Guzman, 09; SDG Chung, C., Fu, 12).
  - Necessary conditions for superconvergence (C., Qiu, Shi, 12, 13, 14).
  - Theory of M-decompositions + new mixed methods (C., Fu, Qiu, Sayas, 16, 17).
- New stabilization functions (Lehrenfeld, Schöberl, 10; Oikawa, 14; HHO Di Pietro, Ern, Lemaire, 14).
- Different formulations of the same method (C. 16).
- Different characterizations leading to the same scheme (C., 16).
- Applications to a wide variety of PDEs.

Ongoing work and open problems

- A posteriori error estimates: Only in terms of $u_h - \widehat{u}_h$ and τ?
- Efficient solvers: Domain decomposition methods?
- Stokes flow: Superconvergence with other formulations?
- Solid mechanics: Optimal convergence for all variables?
- Are there HDG methods which conserve energy?
- Linear transport: Which unknowns superconverge?
- HDG methods for KdV equations: Superconvergence?
- Nonlinear hyperbolic conservation laws: New ways to deal with shocks?

References

- Static condensation, hybridization and the devising of the HDG methods. (48p.)
- The Discontinuous Galerkin methods for fluid dynamics. (111 pp.)
- HDG methods for hyperbolic problems. (20 pp., with N.C.Nguyen and J. Peraire.)